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# Stability Analysis of Linear Systems with Time-varying State and Measurement Delays*

Liang Lu
*State Key Lab of Synthetical Automation for Process Industries*
*Northeastern University, Shenyang, China.*
*liangup@gmail.com*

Pedro Albertos and Pedro García
*Instituto de Automática e Informática Industrial*
*Universidad Politécnica de Valencia*
*Valencia, 46022 Spain*
*{pedro,pggil}@aii.upv.es*

***Abstract*— In this paper, an LMI based analysis approach to check the stability of closed-loop discrete-time linear systems with both time-varying state and measurement delays is presented. The internal as well as the measurement delays are assumed to be time-varying but bounded. The stability analysis procedure allows to determine the delay intervals ensuring the stability of the controlled plant. The procedure is applicable to any plant with two different state delays, as well as to diverse control applications, as illustrated in the reported example. The procedure can be conceptually extended to the general case of systems with multiple time-varying delays.**

## I. INTRODUCTION

The study of time delay systems has been for many years a matter of research. “Time-delay system” is a generic name for a great variety of situations. The simplest problem of constant single input/output delay for SISO plants, since the seminal work of [1], has received much attention and many solutions have been proposed to deal with different kind of plants, unstable and/or non-minimum phase, with long delays (see, for instance, [2] and the references here in). The extension to the case of multiple delays and MIMO plants has been also treated by many researchers, but still there are some open problems in dealing with the interactions and decoupling issues [3], [4], [5].

To consider time-varying delays has been a recent requirement, mainly imposed by the study of network-based control systems, where the transmission delay is random and unknown, [6], but bounded. The problem of internal or state delay has also received a lot of attention, mainly from a theoretical point of view, although any closed-loop controlled plant with input/output delay becomes a plant with internal delay. These delays can be also constant or time-variant, single or multiple. A number of works has been also published with this setting. In the review paper [7], a number of open issues dealing with different delay scenarios are outlined, and the simultaneous presence of delays in the state and/or the input/output is listed. The case of time-variant multiple delays has received a lot of attention, like [8], [9] or [10]. In the work of [11], a deep analysis of the stability and robustness of different processes with internal delays is reported. Most of the works in the literature refer to the stability and robustness analysis, trying to determine the maximum interval of the time delay admissible to keep the system stable [12], [13]. Recently, in the work of [14], a less conservative stability criterion for discrete-time (DT) systems with interval time-varying state delay has been presented, their results being less conservative than those previously reported. The use of a predictor to counteract the effect of time delays, initially introduced in [15], [16] for constant and known delays, was extended to the case of time-variant delays in [17], [18], [19], [20], [21], with promising results. However, this kind of predictor approach is only tractable for input or output delays.

Time delays have been studied for both, continuous and discrete time systems. Although the DT setting allows for an “easy” treatment of the problem by augmenting the state of the plant, for long delays this is not a good solution and the delay should be properly treated [22]. Moreover, if it is time-varying, this approach would lead to a variant state-dimension system. On the other hand, any practical controller will be implemented in a digital computer, thus, a DT version of the solution is needed.

In the rest of the paper, the scenario of discrete-time linear systems with internal time-variant delay systems controlled under a networked setting, that is with both input and output time-varying delays, is considered. This was an “open problem” presented in J. Richard’s review paper [7]. An LMI based stability theorem for discrete-time systems with both state and output time delays is presented. The proof of the theorem is relied on the construction of a type of Lyapunov-Krasovskii functional which by explicitly comparing the bounds of different delays makes the result less conservative. The application of the theorem to an example with different time-varying delay patterns shows the effectiveness of this approach, allowing the comparison with previous published results.

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## II. PROBLEM STATEMENT

Consider the following closed-loop DT linear system with both state and measurement time-varying delays*

$$\begin{aligned} x_{k+1} &= Ax_k + A_d x_{k-d_k} + Bu_k, \quad (1)\\ x_k &= 0, \quad \min\{-d_M, -\tau_M\} \le k \le 0, \\ u_k &= Kx_{k-\tau_k} \quad (2) \end{aligned}$$

where $x_k \in \mathbf{R}^n$ and $u_k \in \mathbf{R}^m$ are respectively the system state and control input, and $K$ in (2) is the state feedback law, the state is assumed to be measurable but subject to an unknown time-varying delay. $A$, $A_d$ and $B$ are the plant parameter matrices with appropriate dimensions, and assumed to be constant and known. The system matrices $A$, $A_d$ do not need to be Hurwitz, and $d_k$ and $\tau_k$ are the time-varying delays satisfying:

$$d_m \le d_k \le d_M, \quad (3)$$
$$\tau_m \le \tau_k \le \tau_M, \quad (4)$$

where $d_m$, $d_M$, $\tau_m$ and $\tau_M$, are known positive integers.

The closed-loop system (1)-(2) can be written as

$$x_{k+1} = Ax_k + A_d x_{k-d_k} + BKx_{k-\tau_k}. \quad (5)$$

As it can be easily shown, the closed-loop system presents two state delays, the first one $d_k$ due to the internal delay and the second one $\tau_k$ due to the delayed state feedback.

Assume that a stabilizing state feedback controller $K$ has been designed under some fixed and known time delays in both, the state and the measurement. The problem is to determine the range of admissible delays, as expressed by (3), (4), ensuring the controlled plant stability under time varying delays. Moreover, if $d_m$ and $\tau_m$ are known, we want to determine the maximum delays $d_M$ and $\tau_M$, to ensure the controlled plant stability under time varying delays within these intervals.

For that purpose, following a well-know approach already used in the literature, a global Lyapunov-Krasovskii function composed by positive definite quadratic functions weighting any pair of delayed signals will be established and the conditions for the stability will be derived by using linear matrix inequalities (LMI).

## III. MAIN RESULT

Next, we will present a sufficient condition for the stability of the closed-loop system (5).

*Theorem 1:* The system (5) is asymptotically stable if there exist positive definite matrices $P$, $Z_1$, $Z_2$, $Z_3$, $Z_4$, $Z_5$, $Z_{dM}$, $Z_{\tau M}$ and positive semi-definite matrices $Q_d$, $Q_{dm}$, $Q_{dM}$, $Q_\tau$, $Q_{\tau m}$, $Q_{\tau M}$ and matrices $S_{d1}$, $S_{d2}$, $T_{d1}$, $T_{d2}$, $S_{\tau 1}$, $S_{\tau 2}$, $T_{\tau 1}$, $T_{\tau 2}$ satisfying the following matrix inequalities,

$$\begin{bmatrix} \Pi & \delta_d S_d & \delta_\tau S_\tau \\ \delta_d S_d^{\mathrm{T}} & -\delta_d Z_{dM} & 0 \\ \delta_\tau S_\tau^{\mathrm{T}} & 0 & -\delta_\tau Z_{\tau M} \end{bmatrix} < 0, \quad (6)$$

$$\begin{bmatrix} \Pi & \delta_d T_d & \delta_\tau T_\tau \\ \delta_d T_d^{\mathrm{T}} & -\delta_d Z_{dM} & 0 \\ \delta_\tau T_\tau^{\mathrm{T}} & 0 & -\delta_\tau Z_{\tau M} \end{bmatrix} < 0. \quad (7)$$

where

$$\begin{aligned} S_d &= \begin{bmatrix} 0 & S_{d1}^{\mathrm{T}} & S_{d2}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \\ T_d &= \begin{bmatrix} 0 & T_{d1}^{\mathrm{T}} & 0 & T_{d2}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \\ S_\tau &= \begin{bmatrix} 0 & 0 & 0 & 0 & S_{\tau 1}^{\mathrm{T}} & S_{\tau 2}^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \\ T_\tau &= \begin{bmatrix} 0 & 0 & 0 & 0 & T_{\tau 1}^{\mathrm{T}} & 0 & T_{\tau 2}^{\mathrm{T}} & 0 & 0 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

$$\Pi = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & 0 & 0 & \Pi_{16} & 0 & \Pi_{18} & \Pi_{1,9} \\ \star & \Pi_{22} & \Pi_{23} & \Pi_{24} & 0 & 0 & 0 & \Pi_{28} & \Pi_{2,9} \\ \star & \star & \Pi_{33} & 0 & 0 & Z_4 & Z_2 & 0 & 0 \\ \star & \star & \star & \Pi_{44} & 0 & Z_3 & Z_1 & 0 & 0 \\ \star & \star & \star & \star & \Pi_{55} & \Pi_{56} & \Pi_{57} & \Pi_{58} & \Pi_{59} \\ \star & \star & \star & \star & \star & \Pi_{66} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \Pi_{77} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -P & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -Z \end{bmatrix}$$

whose elements are

$$\begin{aligned} \Pi_{11} &= -P + (\delta_d+1)Q_d + Q_{dm} + Q_{dM} \\ &\quad + (\delta_\tau+1)Q_\tau + Q_{\tau m} + Q_{\tau M} - Z_5 \\ \Pi_{13} &= \begin{cases} Z_5, & d_m \le \tau_m, \\ 0, & d_m > \tau_m, \end{cases} \\ \Pi_{16} &= \begin{cases} Z_5, & d_m > \tau_m, \\ 0, & d_m \le \tau_m, \end{cases} \\ \Pi_{18} &= A^{\mathrm{T}} P \\ \Pi_{1,9} &= (A-I)^{\mathrm{T}} Z \\ \Pi_{22} &= -Q_d + T_{d1} + T_{d1}^{\mathrm{T}} - S_{d1} - S_{d1}^{\mathrm{T}} \\ \Pi_{23} &= S_{d1} - S_{d2}^{\mathrm{T}} \\ \Pi_{24} &= -T_{d1} + T_{d2}^{\mathrm{T}} \\ \Pi_{28} &= A_d^{\mathrm{T}} P \\ \Pi_{2,9} &= A_d^{\mathrm{T}} Z \\ \Pi_{33} &= -Q_{dm} - \begin{cases} Z_5, & d_m \le \tau_m, \\ 0, & d_m > \tau_m, \end{cases} - Z_4 - Z_2 + S_{d2} + S_{d2}^{\mathrm{T}} \\ \Pi_{44} &= -Q_{dM} - Z_3 - Z_1 - T_{d2} - T_{d2}^{\mathrm{T}} \\ \Pi_{55} &= -Q_\tau + T_{\tau 1} + T_{\tau 1}^{\mathrm{T}} - S_{\tau 1} - S_{\tau 1}^{\mathrm{T}} \\ \Pi_{56} &= S_{\tau 1} - S_{\tau 2}^{\mathrm{T}} \\ \Pi_{57} &= -T_{\tau 1} + T_{\tau 2}^{\mathrm{T}} \\ \Pi_{58} &= K^{\mathrm{T}} B^{\mathrm{T}} P \\ \Pi_{59} &= K^{\mathrm{T}} B^{\mathrm{T}} Z \end{aligned}$$

*We will restrict in this paper to the case where the whole state is accessible and measurable.

$$\Pi_{66} = -Q_{\tau m} - \begin{cases} Z_5, & d_m > \tau_m, \\ 0, & d_m \le \tau_m, \end{cases} - Z_4 - Z_3 + S_{\tau 2} + S_{\tau 2}^{\mathrm{T}}$$
$$\Pi_{77} = -Q_{\tau M} - Z_2 - Z_1 - T_{\tau 2} - T_{\tau 2}^{\mathrm{T}}$$
$$Z = \rho_1^2 Z_5 + (\tau_m - d_m)^2 Z_4 + (\tau_m - d_M)^2 Z_3 + (\tau_M - d_m)^2 Z_2 + (\tau_M - d_M)^2 Z_1 + \delta_d Z_{dM} + \delta_\tau Z_{\tau M}$$

with

$$\rho_1 = \min(\tau_m, d_m), \quad \delta_d = d_M - d_m, \quad \delta_\tau = \tau_M - \tau_m.$$

**Proof.** Due to space limitations, a sketch of the proof is presented. Choose a Lyapunov-Krasovskii functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) \tag{8}$$

with

$$\begin{aligned}
V_1(k) &= x_k^{\mathrm{T}} P x_k \\
V_2(k) &= \sum_{i=k-d_k}^{k-1} x_i^{\mathrm{T}} Q_d x_i + \sum_{i=k-\tau_k}^{k-1} x_i^{\mathrm{T}} Q_\tau x_i \\
V_3(k) &= \sum_{i=k-d_m}^{k-1} x_i^{\mathrm{T}} Q_{d_m} x_i + \sum_{i=k-\tau_m}^{k-1} x_i^{\mathrm{T}} Q_{\tau_m} x_i + \sum_{i=k-d_M}^{k-1} x_i^{\mathrm{T}} Q_{d_M} x_i \\
&\quad + \sum_{i=k-\tau_M}^{k-1} x_i^{\mathrm{T}} Q_{\tau_M} x_i \\
V_4(k) &= \sum_{j=-d_M+1}^{-d_m} \sum_{i=k+j}^{k-1} x_i^{\mathrm{T}} Q_d x_i + \sum_{j=-\tau_M+1}^{-\tau_m} \sum_{i=k+j}^{k-1} x_i^{\mathrm{T}} Q_\tau x_i \\
V_5(k) &= \rho_1 \sum_{i=-\rho_1}^{-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_5 \eta_m \\
&\quad + |\tau_m - d_m| \sum_{i=-\rho_2}^{-\rho_1-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_4 \eta_m \\
&\quad + |\tau_m - d_M| \sum_{i=-\rho_4}^{-\rho_3-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_3 \eta_m \\
&\quad + |\tau_M - d_m| \sum_{i=-\rho_6}^{-\rho_5-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_2 \eta_m \\
&\quad + |\tau_M - d_M| \sum_{i=-\rho_8}^{\rho_7-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_1 \eta_m \\
&\quad + \sum_{i=-d_M}^{-d_m-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_{dM} \eta_m \\
&\quad + \sum_{i=-\tau_M}^{-\tau_m-1} \sum_{m=k+i}^{k-1} \eta_m^{\mathrm{T}} Z_{\tau M} \eta_m
\end{aligned}$$

with

$$\begin{aligned}
\eta_k &= x_{k+1} - x_k, \\
\rho_1 &= \min(\tau_m, d_m), \quad \rho_2 = \max(\tau_m, d_m), \\
\rho_3 &= \min(\tau_m, d_M), \quad \rho_4 = \max(\tau_m, d_M), \\
\rho_5 &= \min(\tau_M, d_m), \quad \rho_6 = \max(\tau_M, d_m), \\
\rho_7 &= \min(\tau_M, d_M), \quad \rho_8 = \max(\tau_M, d_M).
\end{aligned}$$

Compute its increment, $\Delta V(k)$

$$\begin{aligned}
\Delta V_1(k) &= x_{k+1}^{\mathrm{T}} P x_{k+1} - x_k^{\mathrm{T}} P x_k \\
&= [A x_k + A_d x_{k-d_k} + BK x_{k-\tau_k}]^{\mathrm{T}} P [A x_k + A_d x_{k-d_k} \\
&\quad + BK x_{k-\tau_k}] - x_k^{\mathrm{T}} P x_k \\
\Delta V_2(k) &= x_k^{\mathrm{T}} Q_d x_k - x_{k-d_k}^{\mathrm{T}} Q_d x_{k-d_k} + \sum_{i=k-d_{k+1}+1}^{k-d_k} x_i^{\mathrm{T}} Q_d x_i \\
&\quad + x_k^{\mathrm{T}} Q_\tau x_k - x_{k-\tau_k}^{\mathrm{T}} Q_\tau x_{k-\tau_k} + \sum_{i=k-\tau_{k+1}+1}^{k-\tau_k} x_i^{\mathrm{T}} Q_\tau x_i \\
&\le x_k^{\mathrm{T}} Q_d x_k - x_{k-d_k}^{\mathrm{T}} Q_d x_{k-d_k} + \sum_{i=k-d_M+1}^{k-d_m} x_i^{\mathrm{T}} Q_d x_i \\
&\quad + x_k^{\mathrm{T}} Q_\tau x_k - x_{k-\tau_k}^{\mathrm{T}} Q_\tau x_{k-\tau_k} + \sum_{i=k-\tau_M+1}^{k-\tau_m} x_i^{\mathrm{T}} Q_\tau x_i \\
\Delta V_3(k) &= x_k^{\mathrm{T}} (Q_{d_m} + Q_{\tau_m} + Q_{d_M} + Q_{\tau_M}) x_k \\
&\quad - x_{k-d_m}^{\mathrm{T}} Q_{d_m} x_{k-d_m} - x_{k-\tau_m}^{\mathrm{T}} Q_{\tau_m} x_{k-\tau_m} \\
&\quad - x_{k-d_M}^{\mathrm{T}} Q_{d_M} x_{k-d_M} - x_{k-\tau_M}^{\mathrm{T}} Q_{\tau_M} x_{k-\tau_M} \\
\Delta V_4(k) &= (d_M - d_m) x_k^{\mathrm{T}} Q_d x_k + (\tau_M - \tau_m) x_k^{\mathrm{T}} Q_\tau x_k + \\
&\quad \sum_{i=k-d_M+1}^{k-d_m} x_i^{\mathrm{T}} Q_d x_i + \sum_{i=k-\tau_M+1}^{k-\tau_m} x_i^{\mathrm{T}} Q_\tau x_i \\
\Delta V_5(k) &= \eta_k^{\mathrm{T}} (\rho_1^2 Z_5 + (\tau_m - d_m)^2 Z_4 + (\tau_m - d_M)^2 Z_3 \\
&\quad + (\tau_M - d_m)^2 Z_2 + (\tau_M - d_M)^2 Z_1 + \delta_d Z_{dM} + \delta_\tau Z_{\tau M}) \eta_k \\
&\quad - \rho_1 \sum_{i=k-\rho_1}^{k-1} \eta_i^{\mathrm{T}} Z_5 \eta_i - |\tau_m - d_m| \sum_{i=k-\rho_2}^{k-\rho_1-1} \eta_i^{\mathrm{T}} Z_4 \eta_i \\
&\quad - |\tau_m - d_M| \sum_{i=k-\rho_4}^{k-\rho_3-1} \eta_i^{\mathrm{T}} Z_3 \eta_i - |\tau_M - d_m| \sum_{i=k-\rho_6}^{k-\rho_5-1} \eta_i^{\mathrm{T}} Z_2 \eta_i \\
&\quad - |\tau_M - d_M| \sum_{i=k-\rho_8}^{k-\rho_7-1} \eta_i^{\mathrm{T}} Z_1 \eta_i - \sum_{i=k-d_k}^{k-d_m-1} \eta_i^{\mathrm{T}} Z_{dM} \eta_i \\
&\quad - \sum_{i=k-d_M}^{k-d_k-1} \eta_i^{\mathrm{T}} Z_{dM} \eta_i - \sum_{i=k-\tau_k}^{k-\tau_m-1} \eta_i^{\mathrm{T}} Z_{\tau M} \eta_i - \sum_{i=k-\tau_M}^{k-\tau_k-1} \eta_i^{\mathrm{T}} Z_{\tau M} \eta_i
\end{aligned}$$

and establish the requirements to get $\Delta V(k) \le 0$. Denoting by $\delta_d = d_M - d_m$ and $\delta_\tau = \tau_M - \tau_m$, and defining a stacked state vector

$$\mu_k^{\mathrm{T}} = \begin{bmatrix} x_k & x_{k-d_k} & x_{k-d_m} & x_{k-d_M} & x_{k-\tau_k} & x_{k-\tau_m} & x_{k-\tau_M} \end{bmatrix},$$

for some matrices

$$\tilde{S}_d = \begin{bmatrix} 0 & S_{d1}^{\mathrm{T}} & S_{d2}^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$
$$\tilde{T}_d = \begin{bmatrix} 0 & T_{d1}^{\mathrm{T}} & 0 & T_{d2}^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$
$$\tilde{S}_\tau = \begin{bmatrix} 0 & 0 & 0 & 0 & S_{\tau 1}^{\mathrm{T}} & S_{\tau 2}^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}$$
$$\tilde{T}_\tau = \begin{bmatrix} 0 & 0 & 0 & 0 & T_{\tau 1}^{\mathrm{T}} & 0 & T_{\tau 2}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

with appropriate dimensions, we have

$$\begin{aligned}\Delta V(k) \leq \mu^{\mathrm{T}}(k)\Omega\mu(k) + (d_k - d_m)\mu^{\mathrm{T}}(k)\tilde{S}_d Z_{dM}^{-1}\tilde{S}_d^{\mathrm{T}}\mu(k) \\ +(d_M - d_k)\mu^{\mathrm{T}}(k)\tilde{T}_d Z_{dM}^{-1}\tilde{T}_d^{\mathrm{T}}\mu(k) \\ +(\tau_k - \tau_m)\mu^{\mathrm{T}}(k)\tilde{S}_\tau Z_{\tau M}^{-1}\tilde{S}_\tau^{\mathrm{T}}\mu(k) \\ +(\tau_M - \tau_k)\mu^{\mathrm{T}}(k)\tilde{T}_\tau Z_{\tau M}^{-1}\tilde{T}_\tau^{\mathrm{T}}\mu(k) \leq 0\end{aligned}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & 0 & \Omega_{15} & \Omega_{16} & 0 \\ \star & \Omega_{22} & S_{d1} - S_{d2}^{\mathrm{T}} & -T_{d1} + T_{d2}^{\mathrm{T}} & 0 & 0 & 0 \\ \star & \star & \Omega_{33} & 0 & 0 & Z_4 & Z_2 \\ \star & \star & \star & \Omega_{44} & 0 & Z_3 & Z_1 \\ \star & \star & \star & \star & \Omega_{55} & S_{\tau1} - S_{\tau2}^{\mathrm{T}} & -T_{\tau1} + T_{\tau2}^{\mathrm{T}} \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 \\ \star & \star & \star & \star & \star & \star & \Omega_{77} \end{bmatrix}$$

and

$$\begin{aligned}
\Omega_{11} &= A^{\mathrm{T}}PA - P + (\delta_d+1)Q_d + Q_{dm} + Q_{dM} + (\delta_\tau+1)Q_\tau \\
&\quad + Q_{\tau m} + Q_{\tau M} + (A-I)^{\mathrm{T}}Z(A-I) - Z_5 \\
\Omega_{12} &= A^{\mathrm{T}}PA_d + (A-I)^{\mathrm{T}}ZA_d \\
\Omega_{13} &= \begin{cases} Z_5, & d_m \leq \tau_m, \\ 0, & d_m > \tau_m, \end{cases} \\
\Omega_{15} &= A^{\mathrm{T}}PBK + (A-I)^{\mathrm{T}}ZBK \\
\Omega_{16} &= \begin{cases} Z_5, & d_m > \tau_m, \\ 0, & d_m \leq \tau_m, \end{cases} \\
\Omega_{22} &= A_d^{\mathrm{T}}PA_d - Q_d + A_d^{\mathrm{T}}ZA_d + T_{d1} + T_{d1}^{\mathrm{T}} - S_{d1} - S_{d1}^{\mathrm{T}} \\
\Omega_{33} &= -Q_{dm} - \begin{cases} Z_5, & d_m \leq \tau_m, \\ 0, & d_m > \tau_m, \end{cases} - Z_4 - Z_2 + S_{d2} + S_{d2}^{\mathrm{T}} \\
\Omega_{44} &= -Q_{dM} - Z_3 - Z_1 - T_{d2} - T_{d2}^{\mathrm{T}} \\
\Omega_{55} &= K^{\mathrm{T}}B^{\mathrm{T}}PBK - Q_\tau + K^{\mathrm{T}}B^{\mathrm{T}}ZBK + T_{\tau1} + T_{\tau1}^{\mathrm{T}} - S_{\tau1} - S_{\tau1}^{\mathrm{T}} \\
\Omega_{66} &= -Q_{\tau m} - \begin{cases} Z_5, & d_m > \tau_m, \\ 0, & d_m \leq \tau_m, \end{cases} - Z_4 - Z_3 + S_{\tau2} + S_{\tau2}^{\mathrm{T}} \\
\Omega_{77} &= -Q_{\tau M} - Z_2 - Z_1 - T_{\tau2} - T_{\tau2}^{\mathrm{T}} \\
Z &= \rho_1^2 Z_5 + (\tau_m - d_m)^2 Z_4 + (\tau_m - d_M)^2 Z_3 + (\tau_M - d_m)^2 Z_2 \\
&\quad + (\tau_M - d_M)^2 Z_1 + \delta_d Z_{dM} + \delta_\tau Z_{\tau M}.
\end{aligned}$$

After some proper manipulations and Schur complements, the inequalities (6) and (7) are obtained. □

## IV. ILLUSTRATIVE EXAMPLE

In this section an example taken from the literature, under different time-delay patterns, is evaluated and the results are compared with those reported in the previous proposals.

Let us considered the controlled plant (5), being open-loop unstable. The following parameters are assumed:

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0 \\ 0.15 & 1.08 \end{bmatrix}; \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}; \\ B &= \begin{bmatrix} 0 & 0.01 \end{bmatrix}; \quad u_k = Kx_{k-\tau_k} \end{aligned}$$

and $d_k$ and $\tau_k$ are delays satisfying (3), (4).

For a given state feedback control law, $K$, the stability of the closed-loop system can be determined by exploring the LMI condition in Theorem 1.

In what follows, different scenarios will be considered:

a) Assume that $\tau_k = 0$, and $K = \begin{bmatrix} -10 & -18 \end{bmatrix}$

Then, the closed-loop system will be:

$$x_{k+1} = (A+BK)x_k + A_d x_{k-d_k} \tag{9}$$

where the matrices $(A+BK)$ and $A_d$

$$(A+BK) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}; A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix};$$

are the same as defined in [22], [14] (example 1 in both papers). It can be realized that this system is decoupled in two subsystems ($A$ and $A_d$ are triangular and the upper part of $B$ is null) and it should be more interesting to consider an interactive system. Nevertheless, the same parameters are used in order to compare both approaches. Additional experiments, not included here for brevity, show similar results if coupled plants are considered.

By Theorem 3 in [22], the system (9), is asymptotically stable for $d_m = 2$ and it is shown that the stability is ensured in the interval $2 \leq d_k \leq 13$. It is also stable if the delay is in the interval $12 \leq d_k \leq 17$. For Theorem 1 in this paper, for the intervals $2 \leq d_k \leq 16$, and $12 \leq d_k \leq 20$, the stability is proved.

To illustrate the effectiveness of the approach let us review some simulation results for the first time delay interval. Assume that the initial conditions in the system in the example are $x_k = \{1, -2\}$ for $k \leq 0$. In addition, let assume that the delay $d_k$ changes randomly between $d_m = 2$ and $d_M = 16$, which is shown in figure 1.

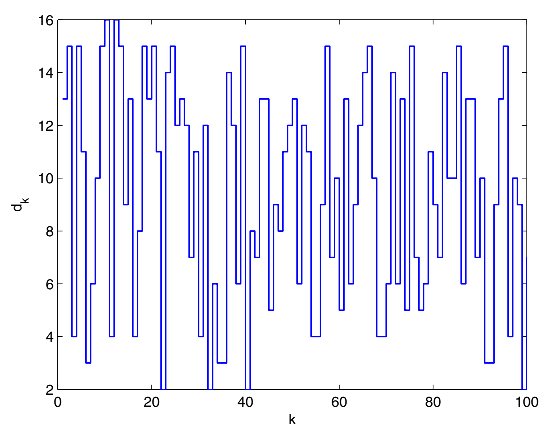


Fig. 1. Time-variant internal delay, $2 \leq d_k \leq 16$.

Then, the state response of the system is given in figure 2. It can be seen from this figure that the system is asymptotically stable. On the other hand, if the time delay $d_k$ will reach larger values ($d_M > 17$) the system stability cannot be guaranteed and, in fact, for some time-variant delay pattern the system becomes unstable. Thus, our results are less conservative.

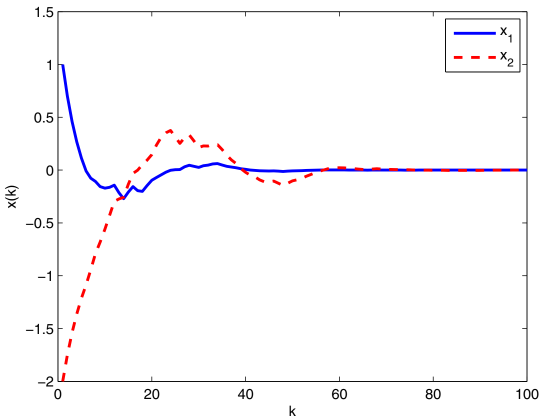


Fig. 2. State evolution from initial conditions.

Note that the stability condition in this paper is quite similar to that previously stated in the literature, but the main novelty in the proposed theorem is that it can be used to evaluate (explore) other scenarios by using the same stability condition (that is, the same theorem is suitable to check different scenarios), being applicable to simultaneously time varying delays in both, the state and the measurement.

For larger delays, and compared with the proposal in Theorem 1 in [14], for $30 \leq d_k \leq 35$, our results are similar.

Again we can illustrate the effectiveness of the approach. Assume that the initial condition in the system in the example are $x_k = \{1, -2\}$ for $k \leq 0$. In addition, let assume that the delay $d_k$ changes randomly between $d_m = 30$ and $d_M = 35$, which is shown in Fig. 3 Then, the state response of the system is given in Fig. 4. It can be seen from this figure that the system is asymptotically stable, but one of the states is highly oscillatory. If the time delay $d_k$ will reach larger values ($d_M = 36$) the system stability will not be guaranteed.

b) Let us assume that $d_k = \tau_k$. Then, the closed-loop systems is:

$$x_{k+1} = Ax_k + (A_d + BK)x_{k-d_k} \tag{10}$$

with a single time delay. Again, the stability can be obtained by exploring the LMI condition in Theorem 1. For example, for the same closed-loop system (5) and the same controlled plant parameters $A$, $A_d$, $B$ and $K$, with $d_m = 1$, an upper bound of $d_M = 4$ is obtained. The simulation results, for $1 \leq d_k \leq 4$ are plotted in Figure 5 with

$$x_k = \{1, -2\}, \quad -4 \leq k \leq 0 \tag{11}$$

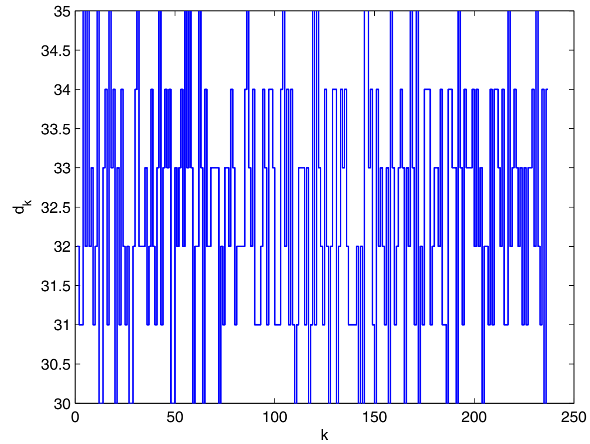


Fig. 3. Time-variant internal delay, $30 \leq d_k \leq 35$.

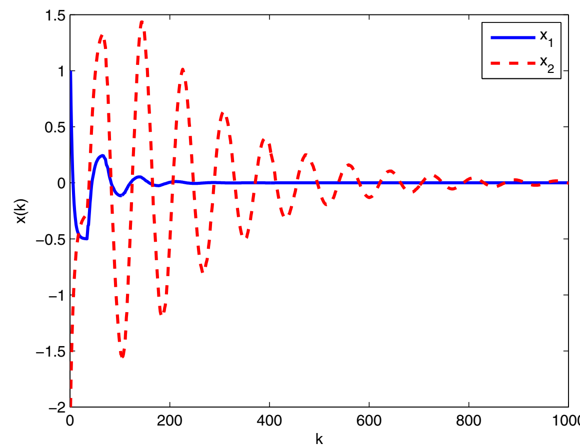


Fig. 4. State evolution from initial conditions.

Note that even the $A$-matrix is not Hurwitz, the closed-loop system (1), in (10), is stable, as a result of the Theorem 1.

c) In the general case, when $d_k \neq \tau_k$, the closed-loop system being:

$$x_{k+1} = Ax_k + A_d x_{k-d_k} + BKx_{k-\tau_k} \tag{12}$$

it is also possible to obtained an upper bound for the delay range. For example, for $1 \leq d_k \leq 6$, the stability condition on Theorem 1 is fulfilled for $2 \leq \tau_k \leq 3$.

The simulation results assuming these ranges of time delay variations, for

$$x_k = \{1, -2\}, \quad -6 \leq k \leq 0 \tag{13}$$

are shown in Figure 6.

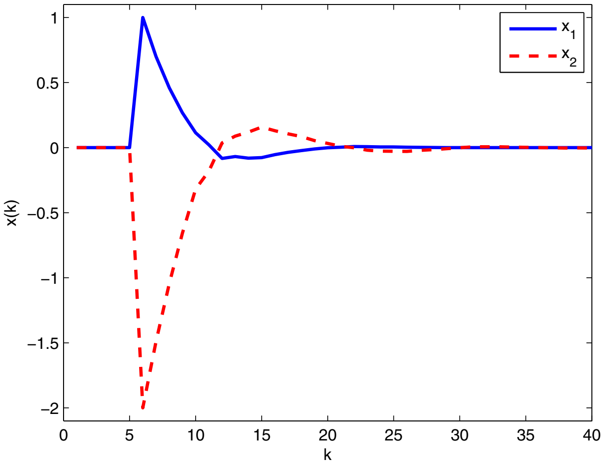


Fig. 5. State response with a common double delay.

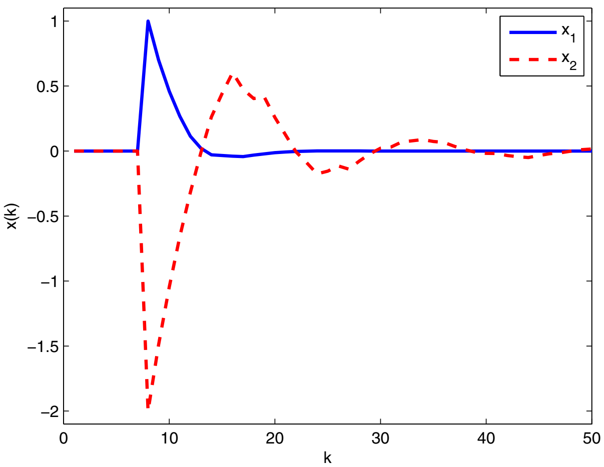


Fig. 6. State and measurement time varying delay, closed-loop state response.

## V. Conclusions

In this paper, an LMI-based general procedure to determine the stability of a plant with two time-varying delays in the state space model has been presented. The procedure has been compared with existing results in the literature and the intervals of the delays to keep the system stable has been shown to be less conservative. This LMI can be generalized to investigate about the stability of systems with multiple time-varying delays. This approach can be also used to analyze the closed-loop stability of delayed state feedback control of a system with internal state delay.

Some simulation results on different scenarios illustrate the applicability of this procedure.

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